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Total domination in partitioned trees and partitioned graphs with minimum degree two

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Abstract Let $G = (V, E)$ be a graph and let $S \subseteq V$. A set of vertices in G totally dominates S if every vertex in S is adjacent to some vertex of that set. The least number of vertices needed in G to totally dominate S is denoted by $\gamma_t(G, S)$. When $S = V$, $\gamma_t(G, V)$ is the well studied total domination number $\gamma_t(G)$. We wish to maximize the sum $\gamma_t(G)+\gamma_t(G, V_1)+\gamma_t(G, V_2)$ over all possible partitions V_1, V_2 of V. We call this maximum sum $f_t(G)$. For a graph H, we denote by $H \circ P_2$ the graph obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex-disjoint. We show that if G is a tree of order $n \geq 4$ and $G \notin \{P_5, P_6, P_7, P_{10}, P_{14}\}$, then $f_t(G) \leq 14n/9$ with equality if and only if $G \in \{P_9, P_{18}\}$ or $G = (T \circ P_2) \circ P_2$ for some tree T. If G is a connected graph of order n with minimum degree at least two, we establish that $f_t(G) \leq 3n/2$ with equality if and only if G is a cycle of order congruent to zero modulo 4.

Keywords Partitioned graphs · Total domination

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1 Introduction

In this paper, we continue the study of the concept of partitions and domination in graphs introduced by Hartnell and Vestergaard [3], and studied, for example, in [7–9]. Here we

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study partitions and total domination in graphs. Throughout this article, only undirected simple graphs without loops or multiple edges are considered.

For notation and graph theory terminology we in general follow [1,4]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order $n = |V|$ and edge set E of size $m = |E|$, and with no isolated vertices. For sets $S, T \subseteq V$, S *totally dominates* T if every vertex in T is adjacent to some vertex of S. If S totally dominates V, then S is called a *total dominating set*, denoted TDS, of G. Every graph without isolated vertices has a TDS, since $S = V$ is such a set. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a TDS. For $U \subseteq V$, we let $\gamma_t(G, U)$ denote the minimum cardinality of a set of vertices in G that totally dominates U. Hence, $\gamma_t(G, V) = \gamma_t(G)$. If $U = \emptyset$, we define $\gamma_t(G, U) = 0$. A set of cardinality $\gamma_t(G, U)$ that totally dominates U in G we call a $\gamma_t(G, U)$-set. If $U = V$, we also call a $\gamma_t(G, U)$-set a $\gamma_t(G)$-set. Total domination in graphs was introduced by Cockayne et al. [2] and is now well studied in graph theory. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [4,5].

By a *partition* of the vertices of a graph $G = (V, E)$, we mean two subsets V_1, V_2 of V with $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$; $\{V_1, V_2\} = \{\emptyset, V\}$ is permitted. Given a partition $\mathcal{P} = \{V_1, V_2\}$ of V, we define the *label* of a vertex v in $\mathcal{P}$, denoted $\ell_{\mathcal{P}}(v)$, as the number $i \in \{1, 2\}$ such that $v \in V_i$. For a graph G, and a partition V_1, V_2 of V, we define $g_t(G; V_1, V_2)$ and $f_t(G; V_1, V_2)$ by

$$g_t(G; V_1, V_2) = \gamma_t(G, V_1) + \gamma_t(G, V_2),$$
$$f_t(G; V_1, V_2) = \gamma_t(G) + g_t(G; V_1, V_2),$$

and $g_t(G)$ and $f_t(G)$ by

$$g_t(G) = \max\{g_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\},$$
$$f_t(G) = \max\{f_t(G; V_1, V_2) \mid V_1, V_2 \text{ is a partition of } V\}.$$

Our aim in this paper is twofold. We wish to establish a sharp upper bound for the function $f_t(G)$ in terms of the order n of a graph G in two cases. First we establish an upper bound for $f_t(G)$ in the case when G is a tree of order at least 4. Second we establish an upper bound for $f_t(G)$ in the case when G is a connected graph with minimum degree at least two. In both cases we characterize the graphs achieving equality in these bounds.

1.1 Notation

Let $G = (V, E)$ be a graph and let $v \in V$ and $S \subseteq V$. The *open neighborhood* of v in G is $N(v) = \{u \in V \mid uv \in E\}$, while the *open neighborhood* of S is the set $N(S) = \cup_{v \in S} N(v)$. Hence for a set $U \subseteq V$, the set S *totally dominates* U if $U \subseteq N(S)$. For a set $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. A vertex of degree k we call a *degree-k vertex*. A degree-1 vertex we call a *leaf* (or an end-vertex), and a vertex adjacent to a leaf we call a *support vertex*. The minimum (resp., maximum) degree among the vertices of G is denoted by $\delta(G)$ (resp., $\Delta(G)$). For disjoint subsets S and T of vertices, we denote by $[S, T]$ the set of edges of G with one end in S and the other in T.

A subset S of vertices in a graph G is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint, i.e., no two vertices from S have a common neighbor, but they may be adjacent.

A set M of edges of G is a *matching* if no two edges in M are incident to the same vertex. A *perfect matching* in G is a matching with the property that every vertex is incident with an edge of the matching.

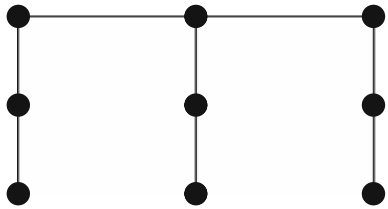

Fig. 1 The tree $(K_1 \circ P_2) \circ P_2$

A cycle on $n \geq 3$ vertices is denoted by C_n and a path on $n \geq 1$ vertices by P_n. A path P_1 is called a trivial path. For $r \geq 3$ and $s \geq 1$, we denote by $L_{r,s}$ the graph obtained by joining with an edge a vertex in C_r to an end-vertex of P_s. We call the graph $L_{r,s}$ a *key*.

For a graph H, we denote by $H \circ P_2$ the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex-disjoint. The graph $H \circ P_2$ is also called the 2-*corona* of H.

2 Known results

In this section, we mention the previous best known upper bounds for $f_t(G)$ when G is a tree of order at least 3 and when G is a connected graph with minimum degree at least two.

Let $G = (V, E)$ be a graph and let $S \subseteq V$. Every minimum TDS in G totally dominates the set S. Hence, $\gamma_t(G, S) \leq \gamma_t(G)$. This implies that $f_t(G) \leq 3\gamma_t(G)$. When G is a tree of order $n \geq 3$, then Cockayne, Dawes, and Hedetniemi [2] showed that $\gamma_t(G) \leq 2n/3$. When G is a connected graph of order n with $\delta(G) \geq 2$, and $G \notin \{C_3, C_5, C_6, C_{10}\}$, then it is shown in [6] that $\gamma_t(G) \leq 4n/7$. Hence the following two results are immediate consequences of known upper bounds on the total domination number of a graph.

Fact 1 ([2]) *If T is a tree of order $n \geq 3$, then $f_t(G) \leq 2n$.*

Fact 2 ([6]) *If $G \notin \{C_3, C_5, C_6, C_{10}\}$ is a connected graph of order n with $\delta(G) \geq 2$, then $f_t(G) \leq 12n/7$.*

3 Main results

We shall prove:

Theorem 1 *If T is a tree of order $n \geq 4$ and $T \notin \{P_5, P_6, P_7, P_{10}, P_{14}\}$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$ or $T = (T' \circ P_2) \circ P_2$ for some tree T'.*

The tree $(K_1 \circ P_2) \circ P_2$, for example, is shown in Fig. 1.

Theorem 2 *If G is a connected graph of order n with $\delta(G) \geq 2$, then $f_t(G) \leq 3n/2$ with equality if and only if $G \cong C_n$ where $n \equiv 0 (\text{mod } 4)$.*

4 Proof of Theorem 1

4.1 Preliminary results

The total domination number of a cycle C_n or a path P_n on $n \geq 3$ vertices is easy to compute.

Lemma 1 ([6]) *For* $n \geq 3$, $\gamma_t(P_n) = \gamma_t(C_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.

Thus for $G \in \{P_n, C_n\}$, if $n \geq 3$ is odd, then $\gamma_t(G) = (n+1)/2$ and if n is congruent to zero modulo 4, then $\gamma_t(G) = n/2$. Finally if n is congruent to two modulo 4, then $\gamma_t(G) = (n+2)/2$.

The total domination number of a key $L_{r,s}$ of order (and size) $r+s$ was determined in [6]. As a consequence of this result, we have the following upper bound on $\gamma_t(L_{r,s})$.

Lemma 2 ([6]) *For* $r \geq 3$ *and* $s \geq 1$, *if* G *is a key* $L_{r,s}$ *of order* $n = r + s$, *then* $\gamma_t(G) \leq (n+2)/2$ *with equality if and only if* $r \equiv 2 (\text{mod} 4)$ *and* $s \equiv 0 (\text{mod} 4)$.

The following lemmas follow immediately from the definitions of $f_t(G)$ and $g_t(G)$.

Lemma 3 *If* G' *is a spanning subgraph of a graph* G *with* $\delta(G') \geq 1$, *then* $g_t(G) \leq g_t(G')$.

Lemma 4 *If* G *is a graph with no isolated vertex, then* $f_t(G) = \gamma_t(G) + g_t(G)$.

We shall use the obvious observation that for a graph G with induced subgraphs G_1, G_2 having no isolated vertices and satisfying $V(G) = V(G_1) \cup V(G_2)$, we have that

$$\gamma_t(G) \leq \gamma_t(G_1) + \gamma_t(G_2),$$
$$g_t(G) \leq g_t(G_1) + g_t(G_2),$$
$$f_t(G) \leq f_t(G_1) + f_t(G_2).$$

The following lemma follows readily from the definition of an open packing.

Lemma 5 *Let* $G = (V, E)$ *be a path* $v_1, v_2, \ldots, v_n$ *of order* n, *and let* V_1, V_2 *be a partition of* V. *If both* V_1 *and* V_2 *are open packings in* G, *then the labels of* $V(P_n)$ *come in alternating pairs but the beginning and the end may be a pair or a single label. More precisely, renaming the sets* V_1 *and* V_2 *if necessary, we have*

$$V_1 = \left(\bigcup_{i=0}^{\lfloor (n-1)/4 \rfloor} \{v_{4i+1}\} \right) \cup \left(\bigcup_{i=0}^{\lfloor (n-2)/4 \rfloor} \{v_{4i+2}\} \right)$$

or

$$V_1 = \left(\bigcup_{i=0}^{\lfloor (n-1)/4 \rfloor} \{v_{4i+1}\} \right) \cup \left(\bigcup_{i=0}^{\lfloor (n-4)/4 \rfloor} \{v_{4(i+1)}\} \right),$$

with the remaining vertices in V_2.

Definition 1 For a graph $G = (V, E)$, we define a partition V_1, V_2 of V to be a good partition if both V_1 and V_2 are open packings in G.

The following lemmas will prove to be useful when proving our main results.

Lemma 6 *Let $G = (V, E)$ be a graph of order $n \geq 2$ with no isolated vertices, and let V_1, V_2 be a partition of V. Then, V_1, V_2 is a good partition of V if and only if $\gamma_t(G, V_1) + \gamma_t(G, V_2) = n$.*

Proof Suppose that V_1, V_2 is a good partition of V. Then for $i \in \{1, 2\}$, no two vertices from V_i can be dominated by a common vertex, and so $\gamma_t(G, V_1) + \gamma_t(G, V_2) = |V_1| + |V_2| = n$. This establishes the necessity. To prove the sufficiency, suppose that V_1, V_2 is not a good partition of V. We may assume that V_1 is not an open packing in G. Thus there exist two vertices in V_1 that have a common neighbor, implying that $\gamma_t(G, V_1) \leq |V_1| - 1$. Hence since $\gamma_t(G, V_2) \leq |V_2|$, we have that $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. □

Lemma 7 *For $n \geq 2$, $g_t(P_n) = n$ and $f_t(P_n) = \lfloor 3n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor$.*

Proof Since every path has a good partition of its vertex set, we have by Lemma 6 that $g_t(P_n) = n$. The desired result now follows from Lemmas 1 and 4. □

Thus by Lemma 7, if $n \geq 3$ is odd, then $f_t(P_n) = (3n + 1)/2$; if $n \equiv 0 (\text{mod } 4)$, then $f_t(P_n) = 3n/2$; if $n \equiv 2 (\text{mod } 4)$, then $f_t(P_n) = (3n + 2)/2$.

Lemma 8 *If $G = (V, E)$ is a path of order $n \geq 2$, and V_1, V_2 is not a good partition of V, then $f_t(G; V_1, V_2) \leq 3n/2$ with strict inequality if $n \not\equiv 2 \,(\text{mod } 4)$.*

Proof By Lemma 6, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. By Lemma 1, $\gamma_t(G) \leq (n + 2)/2$ with strict inequality if $n \not\equiv 2 \,(\text{mod } 4)$. Hence, $f_t(G; V_1, V_2) \leq 3n/2$ with strict inequality if $n \not\equiv 2 \,(\text{mod } 4)$. □

The following lemma is an immediate consequence of Lemma 8.

Lemma 9 *If $G = (V, E)$ is a path of order $n \geq 2$, and V_1, V_2 is a partition of V for which $f_t(G; V_1, V_2) > 3n/2$, then V_1, V_2 is a good partition of V.*

Lemma 10 *If G is a graph of order n without isolated vertices and $S \subseteq V(G)$, then $g_t(G) \leq n + 2|S| - |N(S)|$.*

Proof Let $G = (V, E)$ and let V_1, V_2 be a partition of V. Let $i \in \{1, 2\}$. For each vertex $v \in V_i \setminus N(S)$, we choose an adjacent vertex and call the resulting set of such vertices S'_i. Then, $S \cup S'_i$ totally dominates V_i in G, and so $\gamma_t(G, V_i) \leq |S| + |S'_i|$. Thus, $g_t(G; V_1, V_2) \leq 2|S| + |S'_1| + |S'_2| \leq 2|S| + |V \setminus N(S)| = n + 2|S| - |N(S)|$. Thus for every partition V_1, V_2 of V, $g_t(G; V_1, V_2) \leq n + 2|S| - |N(S)|$. Therefore, $g_t(G) \leq n + 2|S| - |N(S)|$. □

As a special case of Lemma 10, we have the following result.

Lemma 11 *If G is a graph of order n with no isolated vertex and maximum degree at least 3, then $g_t(G) \leq n - 1$.*

Proof Let v be a vertex of maximum degree at least 3 and let $S = \{v\}$. Then, $|S| = 1$ and $|N(S)| \geq 3$, and so the desired result follows from Lemma 10. □

Lemma 12 *If T is a graph of order n that can be obtained from a star on at least four vertices by subdividing some (including the possibility of none) of the edges exactly once, then $f_t(T) < 3n/2$.*

Proof For integers $r \geq k \geq 0$ with $r \geq 3$, let $T = (V, E)$ be obtained from a star $K_{1,r}$ by subdividing k edges exactly once. If $k = 0$, then $n = r + 1 \geq 4$ and $f_t(T) \leq 5 < 3n/2$. Hence we may assume that $k \geq 1$. Then, $\gamma_t(T) = k+1$. Let V_1, V_2 be a partition of V. Then, $\gamma_t(T, V_1) + \gamma_t(T, V_2) \leq k + 3$, and so $f_t(T; V_1, V_2) \leq 2k + 4$. Since $r \geq k$ and $r \geq 3$, we have $3n/2 = 3(k + r + 1)/2 = (3k + r)/2 + r + 3/2 \geq 2k + 9/2$. Thus for every partition V_1, V_2 of V, $f_t(T; V_1, V_2) < 3n/2$. Therefore, $f_t(T) < 3n/2$. □

Next we define a special set $\mathcal{S}$ of small paths.

Definition 2 Let $\mathcal{S} = \{P_1, P_2, P_3, P_5, P_6, P_7, P_{10}, P_{14}\}$.

As a consequence of the remark after Lemma 7 we have the following result.

Lemma 13 *If $T \in \mathcal{S}$ has order $n \geq 2$, then $f_t(T) = (3n + 1)/2$ if n is odd; otherwise, $f_t(T) = (3n + 2)/2$.*

A proof of the following lemma is a simple exercise and is omitted.

Lemma 14 *Let $T = (V, E)$ be a path in $\mathcal{S}$. If $|V| \geq 2$ and $v \in V$ is neither a leaf of a P_5 nor a center of a P_7, then there exists a $\gamma_t(T)$-set containing v.*

Definition 3 Let $\mathcal{T} = \{T \mid T = (T' \circ P_2) \circ P_2$ for some tree $T'\}$.

4.2 Proof of Theorem 1

Recall Theorem 1.

Theorem 1 *If $T \notin \mathcal{S}$ is a tree of order $n \geq 4$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$ or $T \in \mathcal{T}$.*

Proof We proceed by induction on n. When $n = 4$, either $T = K_{1,3}$, in which case $f_t(T) = 5$, or $T = P_4$, in which case $f_t(T) = 6$. In both cases, $f_t(T) < 14n/9$. This establishes the base case. For the inductive hypothesis, let $n \geq 5$ and assume that for all trees $T' \notin \mathcal{S}$ of order n', where $4 \leq n' < n$, $f_t(T') \leq 14n'/9$ with equality if and only if $T' \in \{P_9, P_{18}\}$ or $T' \in \mathcal{T}$.

So let $T = (V, E)$ be a tree of order n with $T \notin \mathcal{S}$. The following observation follows from Lemma 1.

Observation 1 *If $T = P_n$, then $f_t(T) \leq 14n/9$ with equality if and only if $T \in \{P_9, P_{18}\}$.*

By Observation 1, we may assume that T is not a path, for otherwise the desired result follows. With this assumption, we have the following observation by Lemma 11.

Observation 2 $g_t(T) \leq n - 1$.

Observation 3 *If T contains a path on five vertices with one end a leaf in T and with each internal vertex a degree-2 vertex in T, then $f_t(T) < 14n/9$.*

Proof Let $P\colon v, v_1, v_2, v_3, v_4$ be a path in T where $\deg_T(v_4) = 1$ and $\deg_T(v_i) = 2$ for $i = 1, 2, 3$. Let T_1 and T_2 be the components of $T - vv_1$ containing v and v_1, respectively. Then, T_1 is a tree of order $n_1 = n - 4$, while $T_2 = P_4$, and so $g_t(T_2) = n_2 = 4$ and $f_t(T_2) = 6$. Since T is not a path, $n_1 \geq 3$.

Suppose T_1 is a path. Then, $g_t(T_1) = n_1$ and, by Lemma 1, $f_t(T_1) \leq (3n_1 + 2)/2$. Thus, $g_t(T_1) + g_t(T_2) = n$. By Observation 2, $g_t(T) \leq n - 1$, and so $g_t(T) \leq g_t(T_1) + g_t(T_2) - 1$.

Thus, by Lemmas 3 and 4, $f_t(T) = \gamma_t(T) + g_t(T) \le \gamma_t(T_1) + \gamma_t(T_2) + g_t(T_1) + g_t(T_2) - 1 = f_t(T_1) + f_t(T_2) - 1 \le (3n_1 + 2)/2 + 6 - 1 = 3n/2 < 14n/9$. Hence we may assume that T_1 is not a path. In particular, $T_1 \notin \mathcal{S}$ and $n_1 \ge 4$. Thus, by the inductive hypothesis, $f_t(T) \le f_t(T_1) + f_t(T_2) \le 14n_1/9 + 6 < 14n/9$. □

By Observation 3, we may assume that T contains no path on five vertices with one end a leaf in T and with each internal vertex a degree-2 vertex in T.

Let V_1, V_2 be a partition of V. For each edge $uv \in E$, let T_u and T_v denote the components of $T - uv$ containing u and v, respectively. If $T_u \in \mathcal{S}$, then we orient the edge from u to v, while if $T_v \in \mathcal{S}$, then we orient the edge from v to u. (Possibly an edge may be oriented in both directions.)

Observation 4 *If an edge of T has no orientation, then $f_t(T) \le 14n/9$ with equality if and only if $T \in \mathcal{T}$.*

Proof Suppose that an edge $uv \in E$ has no orientation. Applying the inductive hypothesis to T_u and T_v, we have that for $x \in \{u, v\}$, $f_t(T_x) \le 14|V(T_x)|/9$ with equality if and only if $T_x \in \{P_9, P_{18}\}$ or $T_x \in \mathcal{T}$. Hence, $f_t(T) \le f_t(T_u) + f_t(T_v) \le 14|V(T_u)|/9 + 14|V(T_v)|/9 = 14n/9$. Thus if $f_t(T_x) < 14|V(T_x)|/9$ for some $x \in \{u, v\}$, then $f_t(T) < 14n/9$. Suppose then that for $x \in \{u, v\}$, $f_t(T_x) = 14|V(T_x)|/9$, and so $T_x \in \{P_9, P_{18}\}$ or $T_x \in \mathcal{T}$.

Suppose that one of T_u and T_v, say T_u, is a path. Then, $T_u \in \{P_9, P_{18}\}$ and at least one leaf in T_u is a leaf in T that is the end of a path on five vertices every internal vertex of which has degree 2 in T, contrary to assumption.

Hence both T_u and T_v are in the family $\mathcal{T}$. Let $G \cong (P_1 \circ P_2) \circ P_2$. Then both T_u and T_v have disjoint copies of G as a spanning subgraph. Thus, T has as a spanning subgraph the graph $H = kG$, consisting of k disjoint copies of G, for some integer $k \ge 2$, where u and v belong to different copies of G in H. Hence, $n = 9k$. Let G_u and G_v be the copies of G in H that contain u and v, respectively. Let $T_{uv} = G_u \cup G_v \cup \{uv\}$.

We proceed further with two observations about the graph G. We observe first that $\gamma_t(G) = 6$, while $g_t(G) = |V(G)| - 1 = 8$, and so $f_t(G) = 14 = 14|V(G)|/9$. We observe secondly that for every vertex of G there exists a $\gamma_t(G)$-set containing it and if w is a leaf in G or a support vertex in G, then $\gamma_t(G, V(G) \setminus \{w\}) = \gamma_t(G) - 1$.

Suppose that u is a leaf or a support vertex in G_u. Then it follows from our two earlier observations about the graph G that $\gamma_t(T_{uv}) \le \gamma_t(G_u) + \gamma_t(G_v) - 1$, implying that $\gamma_t(T) \le k\gamma_t(G) - 1 = 6k - 1$. Thus since $g_t(T) \le kg_t(G) = 8k$, we have that $f_t(T) \le 14k - 1 = 14n/9 - 1$. Hence we may assume that u is neither a leaf nor a support vertex in G_u. Similarly, v is neither a leaf nor a support vertex in G_v.

Suppose that u or v is the vertex of degree-3 in G_u or G_v, respectively. Then applying Lemma 10 to the tree T_{uv} with $S = \{u, v\}$ we have that $g_t(T_{uv}) \le |V(G_u)| + |V(G_v)| + 2|S| - |N(S)| \le 18 + 4 - 7 = 15$. Thus, $g_t(T) \le g_t(T_{uv}) + (k-2)g_t(G) \le 8k - 1$ while $\gamma_t(T) \le k\gamma_t(G) = 6k$, and so $f_t(T) \le 14k - 1 = 14n/9 - 1$. Hence we may assume that neither u nor v is the vertex of degree 3 in G_u or G_v, respectively.

If $k = 2$, then $T = (T' \circ P_2) \circ P_2$ where $T' = P_2$ consists of the vertices u and v, whence $T \in \mathcal{T}$. Hence we may assume that $k \ge 3$.

Assume that $F \cup (k-3)G$ is a spanning subgraph of T where $F = P_9 \circ P_2$. Let $v_1, v_2, \ldots, v_9$ be the vertices from the path P_9 in F. Then applying Lemma 10 to the graph F with $S = \{v_2, v_3, v_6, v_7\}$ we obtain $g_t(F) \le 27 + 8 - 12 = 23$. Thus, $g_t(T) \le g_t(F) + (k-3)g_t(G) \le 8k - 1$ while $\gamma_t(T) \le k\gamma_t(G) = 6k$, and so $f_t(T) \le 14k - 1 = 14n/9 - 1$. Hence we may assume that $(P_9 \circ P_2) \cup (k-3)G$ is not a spanning subgraph of T. It follows that the degree of every vertex in $G_u \cup G_v$, different from u and v, is unchanged in T. Thus for $x \in \{u, v\}$, if $T_x = (T'_x \circ P_2) \circ P_2$ for some tree T'_x, then we have that $u \in V(T'_u)$ and

$v \in V(T'_v)$. This implies that $T = (T' \circ P_2) \circ P_2$ where T' is the tree $T'_u \cup T'_v \cup \{uv\}$. Thus, $T \in \mathcal{T}$. Hence we have established that either $f_t(T) < 14n/9$ or $f_t(T) = 14n/9$ and $T \in \mathcal{T}$. □

Observation 5 *If an edge of T is oriented in both directions, then $f_t(T) \leq 14n/9$ with equality if and only if $T = (P_1 \circ P_2) \circ P_2$.*

Proof Suppose that an edge $uv \in E$ is oriented in both directions. Hence both components T_u and T_v of $T - uv$ are contained in $\mathcal{S}$. Since both T_u and T_v are paths, $g_t(T_u) + g_t(T_v) = n$. By Observation 2, $g_t(T) \leq n - 1$, and so $g_t(T) \leq g_t(T_u) + g_t(T_v) - 1$.

Since T is not a path, $\deg_T(u) \geq 3$ or $\deg_T(v) \geq 3$. If both $\deg_T(u) \geq 3$ and $\deg_T(v) \geq 3$, then applying Lemma 10 to the tree T with $S = \{u, v\}$, we have $g_t(T) \leq n - 2 = g_t(T_u) + g_t(T_v) - 2$. Thus since $\gamma_t(T) \leq \gamma_t(T_u) + \gamma_t(T_v)$, we have by Lemma 13 that $f_t(T) \leq f_t(T_u) + f_t(T_v) - 2 \leq (3|V(T_u)| + 2)/2 + (3|V(T_u)| + 2)/2 - 2 = 3n/2 < 14n/9$.

Hence we may assume that either $\deg_T(u) \geq 3$ or $\deg_T(v) \geq 3$, but not both. We may assume that $\deg_T(u) \geq 3$, and so $\deg_T(v) \leq 2$. By our assumption following Observation 3, we have that $T_v \in \{P_1, P_2, P_3\}$.

Suppose $T_v = P_1$, and so $|V(T_u)| = n - 1$. If there is a $\gamma_t(T_u)$-set containing u, then $\gamma_t(T) \leq \gamma_t(T_u)$, implying that $f_t(T) \leq \gamma_t(T_u) + g_t(T) \leq (|V(T_u)| + 2)/2 + n - 1 = (3n - 1)/2 < 14n/9$. On the other hand, if there is no $\gamma_t(T_u)$-set containing u, then, by Lemma 14, $T_u = P_7$ and u is the central vertex of this P_7. But then $n = 8$, $\gamma_t(T) = 5$ and $g_t(T) \leq n - 1 = 7$, implying that $f_t(T) \leq 12 = 3n/2 < 14n/9$. Hence we may assume that $T_v \in \{P_2, P_3\}$.

As observed earlier, $g_t(T) \leq g_t(T_u) + g_t(T_v) - 1$. Thus, $f_t(T) \leq f_t(T_u) + f_t(T_v) - 1$. Hence, by Lemma 13, $f_t(T) \leq (3n + \ell)/2$ where ℓ denotes the number of even components of $T - uv$. If $\ell = 0$, then $f_t(T) \leq 3n/2 < 14n/9$, as desired. Hence we may assume that $\ell \in \{1, 2\}$.

Suppose that $\ell = 1$, and so $f_t(T) \leq (3n + 1)/2$. If $n > 9$, then $f_t(T) < 14n/9$. Hence we may assume that $n \leq 9$. Suppose firstly that $P_v = P_2$ and T_u is of odd order. If $T_u \neq P_7$ or if $T_u = P_7$ but u is not the central vertex of P_u, then there is a $\gamma_t(T_u)$-set containing u, and so $\gamma_t(T) \leq \gamma_t(T_u) + 1$, implying that $f_t(T) \leq \gamma_t(T_u) + 1 + g_t(T) \leq (|V(T_u)| + 1)/2 + 1 + n - 1 < 3n/2 < 14n/9$. Hence we may assume that $T_u = P_7$ and that u is the central vertex of T_u. But then $T = (P_1 \circ P_2) \circ P_2 \in \mathcal{T}$. Suppose secondly that $P_v = P_3$. Then, since $n \leq 9$, $P_u = P_6$. By our assumption following Observation 3, the vertex u is a not a support vertex of P_u. But then again $T = (P_1 \circ P_2) \circ P_2 \in \mathcal{T}$.

Suppose finally that $\ell = 2$. Then, $T_v = P_2$ and $T_u \in \{P_2, P_6, P_{10}, P_{14}\}$. Since there is a $\gamma_t(T_u)$-set containing u, we have $\gamma_t(T) \leq \gamma_t(T_u) + 1$, implying that $f_t(T) \leq \gamma_t(T_u) + 1 + g_t(T) \leq (|V(T_u)| + 2)/2 + 1 + n - 1 = 3n/2 < 14n/9$. Hence we have established that either $f_t(T) < 14n/9$ or $f_t(T) = 14n/9$ and $T = (P_1 \circ P_2) \circ P_2$. That proves Observation 5. □

By Observations 4 and 5, we may assume that every edge of T is oriented in exactly one direction. Since T is a tree, it follows that there exist a vertex v with out-degree zero in this oriented tree. Thus for every edge uv in T, $T_u \in \mathcal{S}$ and $T_v \notin \mathcal{S}$. If v is a leaf and u the support vertex adjacent with v, then $T_v = P_1 \in \mathcal{S}$ in $T - uv$, and so v would have out-degree one in the oriented tree, a contradiction. Hence, $\deg_T(v) \geq 2$.

If every neighbor of v in T has degree at most two we define $I = 0$; otherwise, we define $I = 1$. Applying Lemma 10 to the tree T with $S = \{v\}$, we have $g_t(T) \leq n + 2 - \deg_T(v)$. If $I = 1$, and u is a neighbor of v with $\deg_T(u) \geq 3$, then applying Lemma 10 to the tree T with $S = \{u, v\}$, we have $g_t(T) \leq n + 4 - \deg_T(u) - \deg_T(v) \leq n + 1 - \deg_T(v)$. Hence we have the following observation.

Observation 6 $g_t(T) \le n + 2 - \deg_T(v) - I$.

If v is adjacent only to vertices that are isolated in $T - v$ or leaves of a P_5 in $T - v$ or the central vertices of a P_7 in $T - v$, then we define $J = 1$; otherwise, we define $J = 0$. For a graph G, let $\mathrm{oc}(G)$ denote the number of odd components of G and $\mathrm{ec}(G)$ the number of even components of G, and let $k_2(G)$ denotes the number of P_2-components in G. Then it follows from Lemmas 1 and 14 that

$$\gamma_t(T) \le \frac{n-1}{2} + \mathrm{ec}(T-v) + \frac{\mathrm{oc}(T-v)}{2} + J,$$

and if $k_2(T - v) \ge 1$, then

$$\gamma_t(T) \le \frac{n-1}{2} + \mathrm{ec}(T-v) + \frac{\mathrm{oc}(T-v)}{2} + 1 - k_2(T-v).$$

Hence, by Observation 6 and since $\deg_T(v) = \mathrm{ec}(T - v) + \mathrm{oc}(T - v)$, we have the following two upper bounds on $f_t(T)$.

Observation 7 $f_t(T) \le \dfrac{3n}{2} + \dfrac{3}{2} - \dfrac{\mathrm{oc}(T-v)}{2} - I + J.$

Observation 8 *If* $k_2(T - v) \ge 1$, *then* $f_t(T) \le \dfrac{3n}{2} + \dfrac{5}{2} - \dfrac{\mathrm{oc}(T-v)}{2} - I - k_2(T-v).$

We proceed further with three observations.

Observation 9 *If* $J = 1$, *then* $f_t(T) < 14n/9$.

Proof Suppose $J = 1$. Then $\mathrm{oc}(T - v) = \deg_T(v) \ge 2$. By our assumption following Observation 3 there can be no P_5-component of $T - v$. Hence, v is adjacent only to vertices that are isolated in $T - v$ or to the central vertices of a P_7 in $T - v$. If T is a star, then the result follows from Lemma 12. Hence we may assume that v is adjacent to the central vertex of a P_7 in $T - v$. But then $I = 1$. Thus, by Observation 7, we have that $f_t(T) \le 3n/2 + (3 - \deg_T(v))/2$. If $\deg_T(v) \ge 3$, then $f_t(T) \le 3n/2 < 14n/9$. Hence we may assume that $\deg_T(v) = 2$, and so $f_t(T) \le (3n + 1)/2$.If one component of $T - v$ is P_1 and the other one is P_7 with central vertex u, we have that $T_v = P_2 \in \mathcal{S}$, contradicting the fact that v has out-degree zero in the oriented tree. Hence both components of $T - v$ are P_7-components, and so $n = 15$, whence $f_t(T) \le (3n + 1)/2 < 14n/9$. □

Observation 10 *If* $I = J = 0$, *then* $f_t(T) \le 14n/9$ *with equality if and only if* $T = (P_1 \circ P_2) \circ P_2$.

Proof Suppose $I = J = 0$. Then every neighbor of v in T has degree at most two. By our assumption following Observation 3 every component of $T - v$ is therefore isomorphic to P_1, P_2 and P_3 (and so, $\mathrm{ec}(T - v) = k_2(T - v)$). Since T is not a path, $\deg_T(v) \ge 3$. If $T - v$ has no P_3-component, then by Lemma 12, $f_t(T) < 14n/9$. Hence we may assume that $T - v$ has a P_3-component. If $\mathrm{oc}(T - v) \ge 3$, then by Observation 7, $f_t(T) \le 3n/2 < 14n/9$. Hence we may assume that $\mathrm{oc}(T - v) \le 2$. If $k_2(T - v) \ge 2$, then by Observation 8, $f_t(T) \le 3n/2 < 14n/9$. Hence we may assume that $k_2(T - v) \le 1$. Thus, since $\deg_T(v) \ge 3$, we have that $\mathrm{oc}(T - v) = 2$ and $k_2(T - v) = 1$. Since v has out-degree zero in the oriented tree, there can be no P_1-component in $T - v$. Hence, $T - v$ consists of one P_2-component and two P_3-components and v is adjacent to a leaf in each of these components. Thus, $T = (P_1 \circ P_2) \circ P_2$. □

Observation 11 *If $I = 1$ and $J = 0$, then $f_t(T) < 14n/9$.*

Proof Suppose $I = 1$ and $J = 0$. Then, by Observation 7, $f_t(T) \leq 3n/2+(1-\text{oc}(T-v))/2$. If $\text{oc}(T-v) \geq 1$, then $f_t(T) \leq 3n/2 < 14n/9$. Hence we may assume that $\text{oc}(T-v) = 0$, and so $f_t(T) \leq (3n+1)/2$. If $n \leq 9$, then since v by assumption is adjacent to a vertex u of degree at least 3 in T, it follows that $T - v = P_2 \cup P_6$. But then if we consider the edge uv we have that $T_v = P_3 \in \mathcal{S}$, contradicting the fact that v has out-degree zero in the oriented tree. Hence, $n > 9$, whence $f_t(T) \leq (3n+1)/2 < 14n/9$. □

The proof of Theorem 1 now follows from Observations 9, 10 and 11. □

5 Proof of Theorem 2

5.1 Preliminary results

Lemma 15 *If T is a tree of order n that can be obtained from a path $v_1, \ldots, v_{2k+1}$ on $2k+1$ vertices, where $k \geq 0$, by attaching paths P_1 or P_2 to vertices in $\{v_1, v_3, \ldots, v_{2k+1}\}$ such that $\deg_T v_{2i+1} = 3$ for each $i \in \{0, \ldots, k\}$, then $f_t(T) < 3n/2$.*

Proof We proceed by induction on k. If $k = 0$, then T is a star or a subdivided star and the result follows from Lemma 12 and if $k = 1$, then T is one of six small trees (of orders 7, 8, 9, 9, 10, 11) and the result is straightforward to check. This establishes the base cases. Hence we may assume that $k \geq 2$ and that the result of the lemma is true for all trees that can be obtained from a path on $2k'+1$ vertices where $0 \leq k' < k$. Let T be a tree of order n that can be obtained from a path $v_1, \ldots, v_{2k+1}$ on $2k+1$ vertices by the procedure described in the statement of the lemma.

We now consider the forest $F = T - v_3v_4$. Let F_1 and F_2 be the components of F containing v_3 and v_4, respectively. For $i = 1, 2$, let F_i have order n_i, and so $n = n_1 + n_2$. Then, $F_1 \neq (P_1 \circ P_2) \circ P_2$ and F_1 is a tree with $6 \leq n_1 \leq 9$, with three leaves, one vertex of degree 3, and with the remaining vertices of degree 2. Thus, by Theorem 1, $f_t(F_1) < 14n_1/9$. Hence, since $6 \leq n_1 \leq 9$, $f_t(F_1) \leq \lfloor (14n_1 - 1)/9 \rfloor \leq \lfloor 3n_1/2 \rfloor \leq 3n_1/2$. Applying the inductive hypothesis to the tree F_2, we have $f_t(F_2) < 3n_2/2$. Hence, $f_t(T) \leq f_t(F_1) + f_t(F_2) < 3n/2$. □

Lemma 16 *For $n \geq 3$, $f_t(C_n) \leq 3n/2$ with equality if and only if $n \equiv 0 (\text{mod } 4)$.*

Proof Let $G = C_n$, and let V_1 and V_2 be a partition of $V(G)$ satisfying $f_t(G) = f_t(G; V_1, V_2)$. Suppose that both V_1 and V_2 are open packings in G. Let $i \in \{1, 2\}$. Since no two vertices of V_i have a common neighbor, every vertex in $G[V_i]$ has degree one and the set of edges $[V_1, V_2]$ therefore induces a matching in G. Thus since G is 2-regular, we must have that $|V_1| = |V_2|$, $[V_1, V_2]$ induces a perfect matching in G, and that $G[V_i]$ is K_2 or the disjoint union of copies of K_2. Hence, $n \equiv 0 (\text{mod } 4)$.

If n is odd, then at least one of the sets V_1 and V_2 is not an open packing in G, and so, by Lemma 6, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$. By Lemma 1, $\gamma_t(C_n) = (n+1)/2$ for n odd. Hence, $f_t(G) \leq (3n-1)/2$. Therefore we may assume that n is even.

Suppose $n \equiv 2 (\text{mod } 4)$. Then, by Lemma 1, $\gamma_t(C_n) = (n+2)/2$. If V_1 or V_2 is empty, then $f_t(G) \leq 2\gamma_t(C_n) = n + 2 < 3n/2$ since $n \geq 6$. Suppose $|V_1| = 1$. Then, $G[V_2] = P_{n-1}$, and so $\gamma_t(G, V_2) \leq \gamma_t(G[V_2], V_2) \leq \gamma_t(G[V_2]) = \gamma_t(P_{n-1}) = n/2$, implying that $f_t(G) = \gamma_t(G) + \gamma_t(G, V_1) + \gamma_t(G, V_2) \leq (n+2)/2 + 1 + n/2 = n + 2 < 3n/2$. Hence we may assume that $|V_1| \geq 2$ and $|V_2| \geq 2$.

For $i \in \{1, 2\}$, if there are two adjacent vertices with the same label i, then $\gamma_t(G, V_{3-i}) \leq \gamma_t(P_{n-2}) = (n-2)/2$. Hence if both sets V_1 and V_2 contain adjacent vertices, then $f_t(G) = \gamma_t(G)+\gamma_t(G, V_1)+\gamma_t(G, V_2) \leq (n+2)/2+n-2 = (3n-2)/2$. Thus we may assume that at least one of V_1 and V_2, say V_1, is an independent set. This implies that V_2 is not an open packing, and so $\gamma_t(G, V_2) \leq |V_2|-1$. If V_1 is not an open packing, then $\gamma_t(G, V_1) \leq |V_1|-1$, implying that $f_t(G) \leq (n+2)/2 + |V_1| + |V_2| - 2 = (3n-2)/2$. Hence we may assume that V_1 is both an independent set and an open packing. Thus since the vertices in the set V_1 have disjoint neighborhoods in G, $N(V_1) \subseteq V_2$ and $|N(V_1)| = 2|V_1|$. For each vertex $v \in V_2 \setminus N(V_1)$, we choose an adjacent vertex and call the resulting set of such vertices V_2'. Then, $V_1 \cup V_2'$ totally dominates V_2, and so $\gamma_t(G, V_2) \leq |V_1|+|V_2'| \leq |V_1|+|V_2 \setminus N(V_1)| = |V_1| + |V_2| - |N(V_1)| = |V_2| - |V_1|$. Thus since $\gamma_t(G, V_1) = |V_1|$ and $\gamma_t(G) = (n+2)/2$, we have that $f_t(G) \leq (n+2)/2 + |V_2| \leq (n+2)/2 + n - 2 = (3n-2)/2$. Hence if $n \equiv 2 (\text{mod } 4)$, then $f_t(G) \leq (3n-2)/2 < 3n/2$.

Suppose, finally, that $n \equiv 0 (\text{mod } 4)$. Then, by Lemma 1, $\gamma_t(C_n) = n/2$. Since there is a good partition of $V(G)$ in this case, $g_t(G) = n$, implying that $f_t(G) = 3n/2$. □

Lemma 17 *For $n \geq 3$, let $G = C_n$ where $n \equiv 0 (\text{mod } 4)$, and let V_1, V_2 be a partition of $V(G)$. Then, $f_t(G; V_1, V_2) \leq 3n/2$ with equality if and only if V_1, V_2 is a good partition of $V(G)$.*

Proof By Lemma 16, $f_t(G; V_1, V_2) \leq f_t(G) = 3n/2$. If V_1, V_2 is not a good partition of $V(G)$, then V_1 or V_2 is not an open packing in G, and so, by Lemma 6, $\gamma_t(G, V_1)+\gamma_t(G, V_2) \leq n-1$. Together with Lemma 1, $\gamma_t(G) = n/2$, we obtain $f_t(G; V_1, V_2) \leq 3n/2 - 1$. Conversely, if V_1, V_2 is a good partition of $V(G)$, then both V_1 and V_2 are open packings in G, implying by Lemma 6 that $\gamma_t(G, V_1) + \gamma_t(G, V_2) = n$, whence $f_t(G; V_1, V_2) = 3n/2$. □

Lemma 18 *If G is a graph of order n that can be obtained from a cycle $v_0, v_1, \ldots, v_{2k-1}, v_0$ on $2k$ vertices, where $k \geq 2$, by attaching for each $i \in \{0, 1, \ldots, k-1\}$ a path P_1 or P_2 to v_{2i}, then $f_t(G) < 3n/2$.*

Proof Let $G = (V, E)$. If $k = 2$, then G is one of three graphs (of orders 6, 7 and 8) and the result is straightforward to check. Hence we may assume that $k \geq 3$. Let $i \in \{0, 1, \ldots, k-1\}$ and let F_i and G_i be the components of $G - \{v_{2i-1}v_{2i}, v_{2i+2}v_{2i+3}\}$ containing v_{2i} and v_{2i-1}, respectively (where addition is taken modulo $2k$). Then, F_i is a path of order 5, 6 or 7, while G_i is a tree that can be obtained from a path on $2(k-3)+1$ vertices by the procedure described in the statement of the Lemma 15. By Lemma 15, $f_t(G_i) < 3|V(G_i)|/2$.

Let V_1, V_2 be a partition of V such that $f_t(G; V_1, V_2) = f_t(G)$. For $j = 1, 2$, let $V_{i,j} = V_j \cap V(F_i)$. Suppose that $V_{i,1}, V_{i,2}$ is not a good partition of $V(F_i)$. Then, by Lemma 9, $f_t(F_i; V_{i,1}, V_{i,2}) \leq 3|V(F_i)|/2$. Thus, $f_t(G) = f_t(G; V_1, V_2) \leq f_t(F_i; V_{i,1}, V_{i,2}) + f_t(G_i) < 3|V(F_i)|/2+3|V(G_i)|/2 = 3n/2$. Hence we may assume that $V_{i,1}, V_{i,2}$ is a good partition of $V(F_i)$ for each $i \in \{0, 1, \ldots, k-1\}$, for otherwise the desired result follows.

Suppose that for some $i \in \{0, 1, \ldots, k-1\}$, the small component of $G - v_{2i}$ and the small component of $G - v_{2i+2}$ are isomorphic (either to P_1 or P_2). For notational convenience, we may assume that the small component of $G - v_0$ and the small component of $G - v_2$ are isomorphic. Let T_1 and T_2 be the components of $G - \{v_0v_{2k-1}, v_4v_5\}$ containing v_0 and v_{2k-1}, respectively. Then, T_1 is a tree with three leaves, with one vertex of degree 3, and with the remaining vertices of degree 2. Since T_1 is one of four small trees, and since $V_{i,1}, V_{i,2}$ is a good partition of $V(F_i)$ for every $i \in \{0, 1, \ldots, k-1\}$, and in particular for $i = 0, 1$, it is straightforward to check that $f_t(T_1) \leq 3|V(T_1)|/2$. If $k = 3$, then $V(T_2) = \{v_5\}$ and since there exists a $\gamma_t(T_1)$-set containing v_0, it follows that $f_t(G) \leq f_t(T_1)$

$+1 \leq 3(n-1)/2+1 < 3n/2$. If $k \geq 4$, then by Lemma 15, $f_t(T_2) < 3|V(T_2)|/2$, implying that $f_t(G) \leq f_t(T_1) + f_2(T_2) < 3|V(T_1)|/2 + 3|V(T_2)|/2 = 3n/2$.

Hence we may assume that for every $i \in \{0, 1, \ldots, k-1\}$, the small component of $G - v_{2i}$ and the small component of $G - v_{2i+2}$ are not isomorphic. Thus, k must be even. We may assume that for $i \equiv 0 (\text{mod } 4)$, $G - v_i$ has a component isomorphic to P_2 (and therefore for $i \equiv 2 (\text{mod } 4)$, $G - v_i$ has a component isomorphic to P_1). Let C denote the cycle in G (of order $2k$). Let H be the spanning subgraph of G obtained from G by deleting all edges on C incident with vertices v_i where $i \equiv 0 (\text{mod } 4)$. Then, H is isomorphic to $k/2$ disjoint copies of $P_3 \cup K_{1,3}$. Hence since $f_t(P_3 \cup K_{1,3}) = 10$, it follows that $f_t(G) \leq f_t(H) \leq 10|V(H)|/7 = 10n/7 < 3n/2$. □

5.2 Notation

Before proceeding with a proof of Theorem 2, we introduce some additional notation. We define a vertex as ***small*** if it has degree ≤ 2, and ***large*** if it has degree more than 2. In a graph G, let L denote the set of all its large vertices. Suppose $|L| \geq 1$ and let C be any component of $G - L$; it is a path (possibly, containing only one vertex). If C has only one vertex and that is adjacent to two large vertices, or if C has at least two vertices and the two ends of C are adjacent in G to different large vertices, then we say that C is a ***2-path***. Otherwise, when the ends of C are adjacent to the same large vertex, we say that C is a ***2-handle***.

5.3 Proof of Theorem 2

Recall Theorem 2.

Theorem 2 *If G is a connected graph of order n with $\delta(G) \geq 2$, then $f_t(G) \leq 2n/3$ with equality if and only if $G \cong C_n$ where $n \equiv 0 \,(\text{mod}\, 4)$.*

Proof We proceed by induction on $\ell = n+m$, where m denotes the size of G. Note that $n \geq 3$ and $m \geq 3$, and so $\ell \geq 6$. When $\ell = 6$, the graph G is a 3-cycle and $f_t(G) = 4 < 3n/2$. This establishes the base case. For the inductive hypothesis, let $\ell \geq 7$ and assume for all connected graphs G' of order n' and size m' with $n' + m' < \ell$ and with $\delta(G') \geq 2$ that $f_t(G') \leq 2n'/3$ with equality if and only if $G' \cong C_{n'}$ where $n' \equiv 0 \,(\text{mod}\, 4)$.

So let $G = (V, E)$ be a connected graph of order n and size m with $m + n = \ell$ and with $\delta(G) \geq 2$. Suppose that G contains at least one large vertex. Let L be set of all large vertices of G.

Observation 12 *If L contains two adjacent vertices, then $f_t(G) < 3n/2$.*

Proof Suppose that two large vertices u and v are adjacent. Let $G' = G - uv$. Then, G' is a graph of order $n' = n$ and size $m' = m - 1$ and with $\delta(G') \geq 2$. Applying the inductive hypothesis to every component of G', we have that $f_t(G') \leq 3n'/2 = 3n/2$ with equality if and only if every component of G' is a cycle of order congruent to zero modulo 4. By Lemma 3, $f_t(G) \leq f_t(G') \leq 3n/2$. Thus if $f_t(G') < 3n/2$, then $f_t(G) < 3n/2$. If $f_t(G') = 3n/2$, then every component of G' is a cycle of order congruent to zero modulo 4, and so, by Lemma 1, $\gamma_t(G') = n/2$, whence $\gamma_t(G) \leq n/2$. By Lemma 11, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$ for every partition V_1, V_2 of $V(G)$. Thus, $f_t(G) \leq 3n/2 - 1$. □

By Observation 12, we may assume that L is an independent set (for otherwise, the desired result follows).

Observation 13 *If G contains a path on six vertices each internal vertex of which has degree 2 in G and whose end-vertices are not adjacent, then* $f_t(G) < 3n/2$.

Proof Let u and v be the two end-vertices of a path P on six vertices each internal vertex of which has degree 2. Let G' be the graph obtained from G by removing the four internal vertices of this path and adding the edge uv. Then, G' is a connected graph of order $n' = n - 4$ and size $m' = m - 4$ with $\delta(G') \geq 2$. Applying the inductive hypothesis to G', we have that $f_t(G') \leq 3n'/2 = 3n/2 - 6$ with equality if and only if G' is a cycle of order congruent to zero modulo 4. Since the degree of every large vertex of G remains unchanged in G', $\Delta(G') \geq 3$, implying that $f_t(G') < 3n/2 - 6$.

Let V_1, V_2 be a partition of V, and let P be the path u, u_1, u_2, u_3, u_4, v. Thus, $G' = (G - \{u_1, u_2, u_3, u_4\}) \cup \{uv\}$. Let $i \in \{1, 2\}$ and let $V_i' = V(G') \cap V_i$. Let $U \subseteq V(G')$ and let S' be a minimum set of vertices in G' that totally dominates U in G', and so $|S'| = \gamma_t(G', U)$. If $\{u, v\} \subseteq S'$, let $S = S' \cup \{u_1, u_4\}$. If $\{u, v\} \cap S' = \emptyset$, let $S = S' \cup \{u_2, u_3\}$. If $u \in S'$ and $v \notin S'$, let $S = S' \cup \{u_3, u_4\}$. If $u \notin S'$ and $v \in S'$, let $S = S' \cup \{u_1, u_2\}$. In all cases, $|S| = |S'| + 2$ and S totally dominates $U \cup V(P)$ in G. In particular, if $U = V(G')$, then S' is a $\gamma_t(G')$-set and S is a TDS of G, whence $\gamma_t(G) \leq |S| = |S'| + 2 = \gamma_t(G') + 2$. If $U = V_i'$, then S totally dominates V_i in G, and so $\gamma_t(G, V_i) \leq |S| = |S'| + 2 = \gamma_t(G', V_i') + 2$. Hence, $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 6 \leq f_t(G') + 6 < 3n/2$. Thus for every partition V_1, V_2 of V, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$. □

By Observation 13, we may assume that G contains no path on six vertices each internal vertex of which has degree 2 in G and whose end-vertices are not adjacent. Hence since L is an independent set, we have the observation.

Observation 14 *Every* 2*-path contains at most three vertices, while every* 2*-handle contains at most five vertices.*

Observation 15 *If G contains a degree-*3 *vertex that is adjacent to the ends of a* 2*-handle, then* $f_t(G) < 3n/2$.

Proof Assume that there is a degree-3 vertex v that is adjacent to the ends of a 2-handle C. By Observation 14, $2 \leq |C| \leq 5$. By connectivity there exists a 2-path P with an end adjacent to v. Let u be the other large vertex adjacent with an end of P. By Observation 14, $1 \leq |P| \leq 3$. Let G' be the spanning subgraph of graph obtained from G by removing the edge joining u with an end of P. Let G_u and G_v be the components of G' containing u and v, respectively. Let $|V(G_u)| = n_u$ and $|V(G_v)| = n_v$, and so $n = n_u + n_v$. Now, $\delta(G_u) \geq 2$ while G_v is a key $L_{r,s}$ where $r = |C| + 1$ and $s = |P|$. Hence, $3 \leq r \leq 6$ and $1 \leq s \leq 3$. Thus, by Lemma 2, $\gamma_t(G_v) \leq (n_v + 1)/2$. By Lemma 11, $\gamma_t(G_v, V_1) + \gamma_t(G_v, V_2) \leq n_v - 1$ for every partition V_1, V_2 of $V(G_v)$. Thus, $f_t(G_v) \leq (3n_v - 1)/2$. Applying the inductive hypothesis to the graph G_u, $f_t(G_u) \leq 3n_u/2$. Hence, $f_t(G') = f_t(G_u) + f_t(G_v) \leq (3n - 1)/2$. Thus, by Lemma 3, $f_t(G) \leq f_t(G') < 3n/2$. □

By Observation 15, we may assume that every large vertex in G that is adjacent to the ends of a 2-handle has degree at least 4.

Observation 16 *If G contains a* 2*-handle of order* 2, 4 *or* 5*, then* $f_t(G) < 3n/2$.

Proof Suppose there is a 2-handle C where $|C| = k$ and $k \in \{2, 4, 5\}$. Say its ends have common neighbor $v \in L$. By assumption, $\deg_G v \geq 4$. Let $G' = G - V(C)$. Then, G' is a connected graph of order $n' = n - k$ and size $m' = m - k - 1$ and with $\delta(G') \geq 2$. Applying the inductive hypothesis to G', we have that $f_t(G') \leq 3n'/2 = 3(n-k)/2$ with equality if and only if G' is a cycle of order congruent to zero modulo 4.

Let V_1, V_2 be a partition of V and for $i \in \{1,2\}$, let $V_i' = V(G') \cap V_i$. Let $U \subseteq V'(G)$ and let S' be a minimum set of vertices in G' that totally dominates U in G', and so $|S'| = \gamma_t(G', U)$.

Suppose $k = 2$. Then, $S \cup \{v\}$ totally dominates $U \cup V(C)$ in G. It follows that $\gamma_t(G) \leq \gamma_t(G') + 1$, and for $i \in \{1,2\}, \gamma_t(G, V_i) \leq \gamma_t(G', V_i') + 1$. Hence, $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 3 \leq f_t(G') + 3 \leq 3n/2$. If $f_t(G') < 3(n-2)/2$, then $f_t(G; V_1, V_2) < 3n/2$. If $f_t(G') = 3(n-2)/2$, then G' is a cycle (congruent to zero modulo 4). But then we can choose a $\gamma_t(G')$-set to contain v, implying that $\gamma_t(G) \leq \gamma_t(G')$ and $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 2 \leq f_t(G') + 2 \leq 3n/2 - 1$. Thus for every partition V_1, V_2 of V, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$.

Suppose $k = 4$. Let C be the path v_1, v_2, v_3, v_4. Then, $S \cup \{v_2, v_3\}$ totally dominates $U \cup V(C)$ in G. It follows that $f_t(G; V_1, V_2) \leq f_t(G'; V_1', V_2') + 6 \leq f_t(G') + 6 \leq 3n/2$. If $f_t(G') < 3(n-4)/2$, then $f_t(G; V_1, V_2) < 3n/2$. If $f_t(G') = 3(n-4)/2$, then G' is a cycle of order congruent to zero modulo 4, and so, by Lemma 1, $\gamma_t(G') = n'/2 = (n-4)/2$, whence $\gamma_t(G) \leq n/2$. By Lemma 11, $\gamma_t(G, V_1) + \gamma_t(G, V_2) \leq n - 1$, and so $f_t(G; V_1, V_2) \leq 3n/2 - 1$. Thus for every partition V_1, V_2 of V, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$.

Suppose $k = 5$. Let C be the path v_1, v_2, v_3, v_4, v_5. For $i = 1, 2$, let $W_i = V_i \cap V(C)$. If W_1, W_2 is not a good partition of $V(C)$, then by Lemma 8, $f_t(C; W_1, W_2) \leq 3(k-1)/2 = 7$. Thus, $f_t(G; V_1, V_2) \leq f_t(C; W_1, W_2) + f_t(G'; V_1', V_2') \leq 7 + f_t(G') \leq 7 + 3(n-5)/2 = (3n-1)/2$. On the other hand, suppose that W_1, W_2 is a good partition of $V(C)$. Thus, renaming the sets V_1 and V_2 if necessary, we may assume that $W_1 = \{v_1, v_2, v_5\}$ (that is, the labels of v_1, v_2, v_3, v_4, v_5 are given by 1, 1, 2, 2, 1, respectively). But then $\{v, v_1\}$ totally dominates W_1 in G, $\{v_3, v_4\}$ totally dominates W_2 in G, and $\{v, v_3, v_4\}$ totally dominates $V(C)$ in G. Hence, $f_t(G; V_1, V_2) \leq 7 + f_t(G'; V_1', V_2') \leq 7 + 3(n-5)/2 = (3n-1)/2$. Thus for every partition V_1, V_2 of V, $f_t(G; V_1, V_2) < 3n/2$. Therefore, $f_t(G) < 3n/2$. □

By Observations 14 and 16, we have the observation.

Observation 17 *Every* 2*-handle contains three vertices.*

We now construct a spanning subgraph H of G as follows. First from every 2-handle (of order 3) and every 2-path that contains two or three vertices, we delete exactly one edge (both of whose ends necessarily have degree 2). Thus in the resulting graph, there is no 2-handle and every 2-path, if any, has order 1. We then successively delete an edge that joins the single vertex of a 2-path with a large vertex of degree at least 4 in the graph obtained at each stage until no such edge remains. (Thus if a large vertex in the graph constructed at this stage is adjacent with the vertex of a 2-path, then this large vertex has degree 3.) Finally in the resulting graph, we successively delete two of the three edges incident with every large vertex all of whose neighbors are vertices of 2-paths (of order 1) in the resulting graph at each stage until no such large vertex remains. Let H denote the resulting spanning subgraph of G.

By construction, H has no 2-handle and every 2-path in H, if any, has order 1. Further, every large vertex of H that is adjacent to the vertex of a 2-path has degree 3 and has at least one neighbor (of degree 1 or 2) that is not on any 2-path. (Thus no large vertex is adjacent to the ends of more than two 2-paths.) Each leaf in H is either adjacent to a large vertex of H or is adjacent to a degree-2 vertex that is adjacent to a large vertex of H. It follows that every component H' of the spanning subgraph H of G is isomorphic to one of the graphs described in Lemmas 12, 15 or 18: If H' contains only one large vertex, then H' is one of the graphs described in Lemma 12 (stars with possible subdivisions). If the vertices of H' that belong to 2-paths (of order 1) and their neighbors (the large vertices in H') induce a path in H', then H' is one of the graphs described in Lemma 15 (paths with pendants). If the vertices of H' that belong to 2-paths and their neighbors induce a cycle in H', then H' is one of the graphs described in Lemma 18 (cycles with pendants). Hence by Lemma 3, and by Lemmas 12, 15 or 18, it follows that $f_t(G) \leq f_t(H) < 3n/2$.

Hence we have shown that if G contains at least one large vertex, then $f_t(G) < 3n/2$. If G contains no large vertex, then G is a cycle, and the desired result follows from Lemma 16. □

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References

1. Chartrand, G., Lesniak, L.: Graphs & Digraphs, 3rd edn. Chapman & Hall, London (1996)
2. Cockayne, E.J., Dawes, R.M., Hedetniemi, S.T.: Total domination in graphs. Networks **10**, 211–219 (1980)
3. Hartnell, B.L., Vestergaard, P.D.: Partitions and dominations in a graph. J. Combin. Math. Combin. Comput **46**, 113–128 (2003)
4. Haynes, T.W., Hedetniemi, S.T., Slater, P.J.: Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998)
5. Haynes, T.W., Hedetniemi, S.T., Slater, P.J. (eds.): Domination in Graphs: Advanced Topics. Marcel Dekker, New York (1998)
6. Henning, M.A.: Graphs with large total domination number. J. Graph Theory **35**, 21–45 (2000)
7. Henning, M.A., Vestergaard, P.D.: Domination in partitioned graphs with minimum degree two. Discrete Math **307**, 1115–1135 (2007)
8. Seager, S.M.: Partition dominations of graphs of minimum degree 2. Congress. Numer. **132**, 85–91 (1998)
9. Tuza, Z., Vestergaard, P.D.: Domination in partitioned graph. Discussiones Mathematicae Graph Theory **22**, 199–210 (2002)